TCC Week 4

Maximum and comparison principles

Assume $-\Delta + V$ satisfies (a)-property

Lemma. If $u \in \mathcal{D}^1_V(\Omega)$ then $u^+, u^- \in \mathcal{D}^1_V(\Omega)$

$u^+ = \max\{u, 0\}$, $u^- = \max\{-u, 0\} \geq 0$

$\Rightarrow u = u^+ - u^-$, $u^+(x)\, u^-(x) = 0$

Moreover, $\mathcal{E}_V(u^\pm) \leq \mathcal{E}_V(u)$.

If $u, v \in \mathcal{D}^1_V(\Omega) \Rightarrow u \vee v$, $u \wedge v \in \mathcal{D}^1_V(\Omega)$

$u \vee v = \max\{u, v\}$, $u \wedge v = \min\{u, v\}$

$u \in C^\infty$, $u(x) = \sin(x) \notin C^1$ 

$u \in W^{2,2}$ — also not invariant w.r.t. truncations

Lemma (weak Maximum principle)

Assume $-\Delta + V$ satisfies (λ).

Let $w \in \mathcal{D}'_V(\Omega)$ be a supersolution,

$$-\Delta w + V w \geq 0 \quad \text{in } \Omega$$

$(= f \geq 0)$ for some $f \in \left(\mathcal{D}'_V(\Omega)\right)^*$

$\Rightarrow$ $w \geq 0$ in Ω

Remark: $\not\Leftarrow$! $w \geq 0 \not\Rightarrow f \geq 0$!

$$-\Delta w + Vw \geq 0 \iff \langle w, \varphi \rangle_V \geq 0 \ \forall\, 0 \leq \varphi \in \mathcal{D}_V^1(\Omega)$$

Take $\varphi := w^- \in \mathcal{D}_V^1(\Omega)$.

$$0 \leq \langle w, w^- \rangle = \langle w^+ - w^-, w^- \rangle =$$

$$= \underbrace{\langle w^+, w^- \rangle}_{=0} - \underbrace{\langle w^-, w^- \rangle}_{\underbrace{\|w\|_V^2}_{\geq 0}} \leq 0$$

$$\Rightarrow \|w^-\|_V^2 = 0$$

Lemma (weak Max. Principle in H^1_{loc}-setting)

Assume $-\Delta + V$ satisfies (a).

Let $w \in H^1_{loc}(\Omega) \cap L^1_{loc}(\Omega, V\,dx)$ - supersol.,

(*) $\quad -\Delta w + V w \geq 0$ in Ω

Assume $w^- \in \mathcal{D}^1_V(\Omega)$. Then $w \geq 0$ in Ω.

We want to test the equation by w^-

$$(*) \iff \int \nabla w \nabla \varphi + \int V w \varphi \geq 0 \quad \forall \varphi \in \underbrace{H^1_c}_{\text{compact supp}}(\Omega) \cap L^\infty_c(\Omega$$

We can not use w^- as a test!

Take $(\varphi_n) \subset C_0^\infty(\Omega)$ — approx. sequence
for w^-: $\mathcal{E}(\varphi_n) \to \mathcal{E}(w^-)$, $\|\varphi_n - w^-\|_{\mathcal{D}_v^1} \to 0$

Set $w_n := \varphi_n^+ \wedge w^- \Rightarrow 0 \le w_n \le w^-$,
$w_n \in \mathcal{D}_v^1$, $w_n \in H_c^1$.
$w_n = w^- + (\varphi_n^+ - w^-)^-$. Then
$$\mathcal{E}_v(w^- - w_n) = \mathcal{E}_v((\varphi_n^+ - w^-)^-) \le \mathcal{E}_v(\varphi_n - w^-) \to 0$$
— w_n – approx. sequence for $\mathcal{E}_v$
$w_n \wedge w^+ = 0$ $(\operatorname{supp} w_n \subset \operatorname{supp} w^-)$

$$0 \le \langle w, w_n \rangle_v = -\langle w^-, w_n \rangle_v \to -\mathcal{E}_v(w^-) \le 0$$
$\Rightarrow w^- = 0$ ∎

Corollary (Comparison principle)

Let $w \in H^1_{loc} \cap L^1_{loc}(\Omega, V dx)$ be a supersol.

and $v \in$ ... be a subsol

to $-\Delta u + Vu = 0$ in Ω,

If $(w-v)^- \in \mathcal{D}^1_V(\Omega) \Rightarrow w \geq v$ in Ω

◀ Apply Weak M.P. in H^1_{loc} to $w-v$:

$$-\Delta(w-v) + V(w-v) \geq 0 \text{ in } \Omega$$ ▶

Minimal and large positive solutions

$R>0$

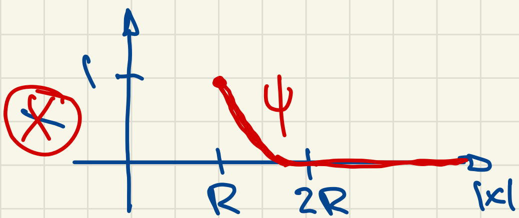
Assume $-\Delta+V$ satisfies (λ) and $\Omega=\mathbb{R}^N\setminus\overline{B}_R$

Lemma. $\exists$ unique "finite energy" solution u_1 to

$$-\Delta u+Vu=0 \text{ in } B_R^c, \quad u \text{ "="} 1 \text{ on } |x|=R$$

(weak sence)

$\blacktriangleleft$ Take smooth ψ: $\psi=1$ on $|x|=R$, $\psi=0$ in $|x|>2R$, $\psi\geq 0$



Set $-\Delta\psi+V\psi=F\in\left(\mathcal{D}^1_V(B_R^c)\right)^*$

exerc.

Solve $-\Delta v+Vv=-F$ in $\mathcal{D}^1_V$ $(*)$

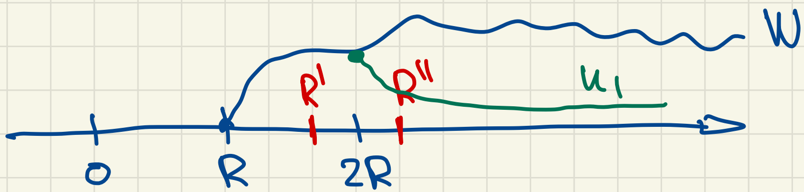
$\exists$ unique v_F that solves $(*)$ and $v_F\in\mathcal{D}^1_V$ $\blacktriangleright$

But then $u_1=v_F+\psi$ satisfies $-\Delta u_1+Vu_1=0$

u_1 in the construction of the Lemma is a Minimal Positive solution to $-\Delta+V$ in B_R^c

Lemma. Assume $-\Delta+V$ satisfies $\geq$
$\forall$ positive supersol. $w>0$ in B_R^c

$$\exists\, c>0 : \quad w \geq c\, u_1 \quad \text{in } B_{2R}^c$$



$$c = \inf_{R'<|x|<R''} w > \int_{R'<|x|<R''} w > 0$$

$w \geq \frac{1}{c} u_1$ on $|x|=R$ $\Longrightarrow$ $w-u_1$ "$=$" 0 on $|x|=R$

$\Longrightarrow (w-u_1)^- \in \mathcal{D}'_V(\Omega)$

$-\Delta(w-u_1)+V(w-u_1) \geq 0$

$w \geq u_1$ in $|x| \geq 2R$

by Comparison

Lemma. Assume $-\Delta + V$ satisfies (2)

Then $\exists$ unique (up to a scalar) positive sol. u_0 to

$-\Delta u + Vu = 0$ in B_R^c, $u = 0$ in $|x| = R$.

◀ Agmon, Th. 3.1. ▶ **Remark.** $u_0 \notin \mathcal{D}_V^1$

Example. $V = 0$, $-\Delta$ in $\mathbb{R}^N \setminus \bar{B}_1$, $N \geq 3$.

$u_1 = |x|^{2-N}$ is the minimal sol: $-\Delta |x|^{2-N} = 0$ in B_1^c

$u_1(1) = 1$. (Ex. – u_1 is "finite energy")

$u_0(x) = 1 - |x|^{2-N}$

Example: $-\Delta$ in $\mathbb{R}^2 \setminus \overline{B_1}$

$-\Delta \log|x| = 0$ in $\mathbb{R}^2 \setminus \overline{B_1}$, $\log|1| = 0$

$\Rightarrow u_0 = \log|x|$

$u_1 = 1$: $-\Delta u_1 = 0$ in B_1^c

$u_1 = 1$ on $|x| = 1$

Exercise: Prove u_1 "$\in$" $\mathcal{D}'_V(B_1^c)$

in the sense that $u_1 = w_1 + \psi$, $w_1 \in \mathcal{D}'_V$

ψ is like in Lemma $*$